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AN IMPROVING APPROACH FOR INTERVAL AND FUZZY NUMBER LINEAR PROGRAMMING PROBLEMS

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Abstract. In this study, Interval number and Fuzzy number Linear Programming problems are addressed. We deal with some convenient methods to solve the addressed problems. We focus on preparing an improving approach based on Tang Shaocheng method to solve linear programming problems with interval or fuzzy numbers. Also, some applications of our proposed approach in fractional programming are presented. Particularly, we propose an algorithm using our improved approach to solve a fuzzy multi-objective linear fractional programming problem. Finally, we illustrate the proposed approach using numerical and some real life examples.

Keywords: Fuzzy number, Fuzzy linear programming, Fractional programming, Interval number, Tong Shaocheng method.

AMS Subject Classification: 90-XX, 90Bxx, 90Cxx.

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1 Introduction

Mathematical programming plays an important role in optimization problems. On the other hand, uncertainty and inaccuracy of data are inseparable from the many real- world phenomena. Hence, most of the time it is difficult to determine the coefficients of a mathematical programming model as a real number. In such a situation, The interval mathematical programming and fuzzy mathematical programming are the powerful alternative tackles to formulate these kind of problems. When some data about distribution of coefficients as fuzzy intervals (here fuzzy optimization problems may be appeared) are available, the method developed can be used as a series of mathematical programming problems on α-levels. In general, interval analysis is the first key component of any fuzzy interval analysis and in particular in fuzzy optimization.

In Shaocheng (1994) the author introduced his approach to solve two kinds of linear programming, namely interval number and fuzzy number linear programming. He converted a Linear Programming problem with Interval numbers (an ILP problem) into the two individual classical linear programming considering maximum value range and minimum value range inequalities as the constraint conditions and then obtained an interval optimal solution to the original problem. In Nasseri et al. (2019) an approach was proposed based on Shaocheng method in which they used the concept of convex intervals and gave an integrated model. In Saati et al. (2002) it was introduced a fuzzy version of CCR model with asymmetrical triangular fuzzy number and

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proposed a procedure to solve it. Also, for acquaintance with the case studies in the interval programming, references Nasseri & Bavandi (2018, 2017); Pal et al. (2012, 2010); Salary Pour Sharif Abad et al. (2020) may be useful.

Some researchers introduced optimality conditions in different types of problems with interval parameters and fuzzy parameters. The Karush-Kuhn-Tucker (KKT) optimality conditions for the optimization (single-objective programming) problem with interval-valued objective function, and for multi-objective programming problems with interval-valued objective functions were investigated in Wu (2007, 2009) (see also Horst et al. (2000)). The optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative was considered by Chalco-Cano et al Chalco-Cano (2013). More recently, optimality condition is studied for generalized differentiable interval-valued functions by Osuna-Gomez et al Abbasi et al. (2021). Furthermore, in Nasseri & Khazaee Kohpar (2015) it was studied Pareto-optimal solutions in multi-objective linear programming with fuzzy numbers in the context of supplier selection problem. Morevover, in Chanas & Kuchta (1996); Hong-yi (2007); Allahdadi et al. (2016) and Wu (2008a,b) some valuable researches was done about interval mathematical programming.

Many real life problems are based on the relative ratios of economic values, such as financial and corporate planning (debt/equity ratio), production planning (inventory/sales, output/employee) and many others. Linear Fractional Programming (LFP) is a powerful tackle to manage these situations Bavandi & Nasseri (2022). In Charnes & Cooper (1962) an LFP problem was solved by considering a variable transformation method and a new objective function method was developed in Bitran & Novaes (1973). In Tantawy (2008, 2007) it was introduced two different approaches, namely a feasible direction method and a duality approach, to solve the LFP problem.

As we have mentioned before, uncertainty and inaccuracy of data occur naturally in many real-world problems, specially in LFP problems. Therefore, the interval and fuzzy parameters and variables will be used in LFP problems, in the normal way. Chakraborty & Gupta (2002) dealt with a solution procedure for a multi-objective linear fractional programming problem. They formulated an equivalent multi-objective linear programming form of the problem, and then used fuzzy mathematical programming to propose their procedure. In Borza et al. (2012) the LFP problem is solved with interval coefficients with the help of an objective function based on the Charnes and Cooper's method (Charnes & Cooper, 1962).

Among current efforts in this erea, we can name the following researches:
Mehlawat & Kumar (2012) proposed a method for computing an (α, β) acceptable optimal value, where α and β, belonged to $[0, 1]$, determine the degrees of satisfaction associated with the fuzzy objective function and with the fuzzy constraints, respectively. Veeramani and Sumathi Tantawy (2014) proposed a method for solving fuzzy linear fractional programming problem by the use of a fuzzy mathematical programming approach. In StanojeviA & StanojeviA (2013) an approach is analyzed for solving a fully fuzzy linear fractional programming problem, where all of the variables and coefficients are triangular fuzzy numbers. Tantawy (2016) proposed a numerical approach to solve an LFP problem in a fuzzy environment.

In this research, we use the approach proposed in Chakraborty & Gupta (2002) and our proposed improved approach to reach a general algorithm to obtain a solution for a multi-objective linear programming problem with fuzzy parameters. In this article, we focus on Tong Shaocheng method and approaches based on that. We instance Nour method as these methods. However, any other procedure which is based on Shaocheng method could be used here. After a comprehensive study on the methods, we have given some advantages, disadvantages and results of the considered methods, and so an improved approach has been obtained. Moreover, we have applied our improved approach to fractional programming, specially to fuzzy multi-objective linear fractional programming.

The rest of this article is organized as follows: in Section 2, we present some applied preliminaries. Interval linear and fuzzy linear programming problems are considered in Section 3. In Section 4, we first introduce the interval linear fractional and fuzzy linear fractional programming problems and then propose a new approach for the discussed models which appear and are formulated from real life situations. In each part, we also present an application of our improved approach in fractional programming to illustrate it. We finally assign Section 5 for the conclusions.

2 Preliminaries

In this section, after introducing the interval numbers and fuzzy numbers, we define some elementary arithmetic operations on these interval numbers and fuzzy numbers. Besides, some basic concepts and definitions are presented as they will be used throughout the study.

Definition 1. *We define an interval number as follows:*

$$A = [\underline{a}, \overline{a}] = \{x \mid \underline{a} \leq x \leq \overline{a}, x \in \mathbb{R}\}$$

where $\underline{a}, \overline{a}$ are the lower and upper bounds of interval number A on the real line $\mathbb{R}$, respectively. It should be stated that, as a closed interval in $\mathbb{R}$ restricts the inaccuracy in data, A is also known as Confidence interval.

2.1 Interval arithmetic

Consider two interval numbers, namely $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$. Now, we define arithmetic operation on these interval numbers as follows (Abbasi et al. (2015)):

(1) $A \oplus B = [\underline{a}, \overline{a}] \oplus [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$,
(2) $\overline{A} = \overline{[\underline{a}, \overline{a}]} = [-\overline{a}, -\underline{a}]$,
(3) $A \ominus B = [\underline{a}, \overline{a}] \ominus [\underline{b}, \overline{b}] = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$,
(4) $A \odot B = [\underline{a}, \overline{a}] \odot [\underline{b}, \overline{b}]$
$= [\min\{\underline{a}\,\underline{b}, \underline{a}\,\overline{b}, \overline{a}\,\underline{b}, \overline{a}\,\overline{b}\}, \max\{\underline{a}\,\underline{b}, \underline{a}\,\overline{b}, \overline{a}\,\underline{b}, \overline{a}\,\overline{b}\}]$
(in the case of considering the intervals on $\mathbb{R}^+$ (nonegative real line), it could be written as $A \odot B = [\underline{a}\,\underline{b}, \overline{a}\,\overline{b}]$),
(5) $k \cdot A = [k, k] \odot [\underline{a}, \overline{a}] = [k\underline{a}, k\overline{a}]$; $k \in \mathbb{R}_+$,
(6) $A^{-1} = [\underline{a}, \overline{a}]^{-1} = [\frac{1}{\overline{a}}, \frac{1}{\underline{a}}]$; $[\underline{a}, \overline{a}] \subseteq \mathbb{R}_+, 0 \notin [\frac{1}{\overline{a}}, \frac{1}{\underline{a}}]$,
(7) $A \oslash B = [\underline{a}, \overline{a}] \oslash [\underline{b}, \overline{b}] = [\underline{a}, \overline{a}] \odot [\frac{1}{\overline{b}}, \frac{1}{\underline{b}}]$
$= [\min\{\frac{\underline{a}}{\overline{b}}, \frac{\underline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}\}, \max\{\frac{\underline{a}}{\overline{b}}, \frac{\underline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}\}]; 0 \notin [\underline{b}, \overline{b}]$
(in the case of considering the intervals on $\mathbb{R}^+$, it could be written as $A \oslash B = [\frac{\underline{a}}{\overline{b}}, \frac{\overline{a}}{\underline{b}}]$),
where the items above represent the addition of two interval numbers, projection of an interval number, subtraction of two interval numbers, multiplication of two interval numbers, multiplication of an interval number by an scalar number, inverse of an interval number and division of two interval numbers, respectively.

Definition 2. *A fuzzy set A for the universal set X is determined by the membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ in $[0, 1]$, represents the membership degree of x in A for all x belongs to X. In general, a fuzzy set A can thus be denoted by a set of ordered pairs of the form $A = \{(x, \mu_A(x)) : x \in X\}$.*
Now, a fuzzy number is a convex normalized fuzzy set of the real line $\mathbb{R}$, whose membership function is piecewise continuous (see for more details Abbasi et al. (2015)).

There are different types of fuzzy numbers in the literature (see Abbasi & Allahviranloo (2019); Abbasi et al. (2021); Abbasi (2019)), among which the triangular fuzzy number (linear fuzzy number) is one of the most frequently used fuzzy numbers in the researches, especially due to the simple linear form of its membership function. That is why in the present research, too, we make use of the triangular fuzzy number.

Definition 3. *A triangular fuzzy number $\tilde{a} = (a^l, a^m, a^u)$ with the triangular membership function $\mu_{\tilde{a}}(x)$ as given in (1), could be represented in Figure 1.*

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a^l}{a^m-a^l}, & a^l \leq x \leq a^m, \\ \frac{x-a^u}{a^m-a^u}, & a^m \leq x \leq a^u, \\ 0, & x < a^l, \quad x > a^u. \end{cases} \tag{1}$$

If $a^m - a^l = a^u - a^m$, then it is called a symmetric triangular fuzzy number and can thus be

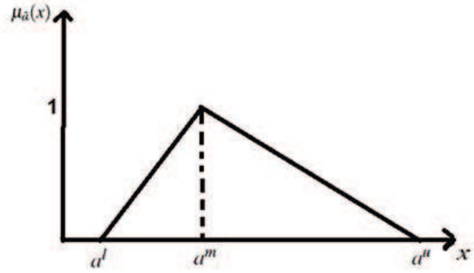

Figure 1: A triangular fuzzy number $\tilde{a}$

introduced with an abbreviated notation of the form $L(a^m, c)$, where $c = a^m - a^l$.

Let $F(\mathbb{R})$ is the set of all fuzzy numbers on $\mathbb{R}$.

Definition 4. *(Abbasi et al. (2015)) Given a fuzzy set $\tilde{a}$, its α- cut, $\tilde{a}_\alpha$, can be defined as follows:*

$$\tilde{a}_\alpha = \{x \in X \mid \mu_{\tilde{a}}(x) \geq \alpha\},$$

where α is the confidence level.

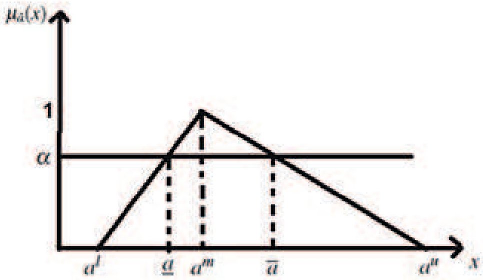

Figure 2: α-cut of triangular fuzzy number $\tilde{a}$

In general, $\tilde{a}_\alpha$ is the crisp set of all elements $x \in X$ that belong to the fuzzy set $\tilde{a}$ at least to the confidence level α belongs to $[0, 1]$. For example, as we can see in Figure 2, the α-cut of triangular fuzzy number $\tilde{a}$ could be considered as $[\underline{a}, \overline{a}]$.

Lemma 1. *If $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, then for $\tilde{a} \in F(\mathbb{R})$, we have $\tilde{a}_{\alpha_2} \subseteq \tilde{a}_{\alpha_1}$.*

Remark 1. *According to the membership function (1), for $\alpha \in [0, 1]$, we have*

$$\tilde{a}_\alpha = [\underline{a}, \overline{a}] = [\alpha a^m + (1-\alpha)a^l, \alpha a^m + (1-\alpha)a^u].$$

2.2 Fuzzy arithmetic

There are two general approaches with respect to arithmetic of fuzzy numbers: the first one is based on minimum extension principle proposed by Zadeh Zadeh (1978) and the second one is to use interval arithmetic on the α- cuts of fuzzy numbers. For details, we refer to Abbasi et al. (2015). In this research, we adopt the second approach which uses decomposition principle, as we mention in the follow observation:
Observation: For $\tilde{a}, \tilde{b} \in F(\mathbb{R})$ with their corresponding α- cuts $\tilde{a}_\alpha = [\underline{a}_\alpha, \overline{a}_\alpha]$, $\tilde{b}_\alpha = [\underline{b}_\alpha, \overline{b}_\alpha]$, an arbitrary arithmetic operation $\tilde{a} * \tilde{b}$ yields a fuzzy number on $\mathbb{R}$ and can be defined as

$$\tilde{a} * \tilde{b} = \cup_\alpha \alpha (\tilde{a} * \tilde{b})_\alpha \quad , \quad (\tilde{a} * \tilde{b})_\alpha = \tilde{a}_\alpha * \tilde{b}_\alpha \quad ; \quad \alpha \in (0, 1].$$

Now, as $\tilde{a}_\alpha$, $\tilde{b}_\alpha$ and $(\tilde{a} * \tilde{b})_\alpha$ for $\alpha \in (0, 1]$ are closed intervals on $\mathbb{R}$, by using interval arithmetic operation (subsection 2.1) on closed intervals $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ with respect to "$*$", we can obtain the closed interval $(\tilde{a} * \tilde{b})_\alpha$. In fact, the arithmetic operation on fuzzy numbers essentially is the same arithmetic operation on their corresponding intervals of α- cuts. Therefore, we can define the elementary fuzzy arithmetic operation as follows:

(1) $\tilde{a}_\alpha \oplus \tilde{b}_\alpha = [\underline{a}_\alpha + \underline{b}_\alpha, \overline{a}_\alpha + \overline{b}_\alpha]$,
(2) $\tilde{a}_\alpha \ominus \tilde{b}_\alpha = [\underline{a}_\alpha - \overline{b}_\alpha, \overline{a}_\alpha - \underline{b}_\alpha]$,

also for $\tilde{a}, \tilde{b}$ on $\mathbb{R}^+$, we have:

(3) $\tilde{a}_\alpha \odot \tilde{b}_\alpha = [\underline{a}_\alpha \, \underline{b}_\alpha, \overline{a}_\alpha \, \overline{b}_\alpha]$,
(4) $\tilde{a}_\alpha \oslash \tilde{b}_\alpha = [\frac{\underline{a}_\alpha}{\overline{b}_\alpha}, \frac{\overline{a}_\alpha}{\underline{b}_\alpha}]; \quad 0 \notin [\underline{b}_\alpha, \overline{b}_\alpha]$,
(5) $(k \cdot \tilde{a})_\alpha = k \cdot \tilde{a}_\alpha = [k\underline{a}_\alpha, k\overline{a}_\alpha]; \; k \in \mathbb{R}^+$.

Here, we propose the approach of Ishibuchi and Tanaka Ishibuchi & Tanaka (2000) for the ranking of two interval numbers.
Suppose that $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ are two interval numbers, then maximum of A and B can be defined by an order relation $\leqslant_{LR}$ between A and B as follows:

$A \leqslant_{LR} B$ iff $\underline{a} \leqslant_{LR} \underline{b}$, $\overline{a} \leqslant_{LR} \overline{b}$,
$A <_{LR} B$ iff $A \leqslant_{LR} B$, $A \neq B$.
Besides, another order relation $\leqslant_{mw}$, where $\leqslant_{LR}$ cannot be applied, can be used to complete the order relation rule.

$A \leqslant_{mw} B$ iff $m(A) \leqslant m(B)$, $w(A) \geqslant w(B)$,
$A <_{mw} B$ iff $A \leqslant_{mw} B$, $A \neq B$,
where $m(A) = \dfrac{1}{2}(\underline{a} + \overline{a})$ is the mid-point and $w(A) = \dfrac{1}{2}(\overline{a} - \underline{a})$ is the half-width of interval $A = [\underline{a}, \overline{a}]$.

3 Interval Linear and Fuzzy Linear Programming problems

In this section, we introduce a Linear Programming problem with Interval numbers and named it in the abbreviated form ILP. Some useful definitions and one theorem are also proposed. Subsection 3.2, deals with a linear programming whose coefficients are fuzzy numbers therein. In other words, an imprecise context, expression such as " about ", " almost " and the like are normally used. For such a problem, it is difficult to come up with an appropriate solution by means of classical linear programming problems. Nevertheless, these kind of problems are often observed in practical optimization.

3.1 Interval Linear Programming problem

We deal with the standard form of Interval Linear Programming problem which is introduced in Shaocheng (1994):

$$\begin{aligned} \min \quad & Z(x) = \sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j] x_j, \\ s.t. \quad & \sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}] x_j \geq [\underline{b}_i, \overline{b}_i], \quad i = 1, \ldots, m, \\ & x_j \geq 0, \quad j = 1, \ldots, n. \end{aligned} \tag{2}$$

The other forms can be converted into the standard form.
According to the operations of interval number Tanaka (1984), each inequality in (2) can be transformed into 2^{n+1} inequalities such as
$a_1x_1 + a_2x_2 + \ldots + a_nx_n \geq b$; $\quad a_j \in \{\underline{a}_{ij}, \overline{a}_{ij}\}, \quad b \in \{\underline{b}_i, \overline{b}_i\}$.
Suppose that D_i stand for the set of solution to the i inequality and

$$\overline{D} = \bigcup_{i=1}^{2n+1} D_i \quad , \quad \underline{D} = \bigcap_{i=1}^{2n+1} D_i \ .$$

Definition 5. *For inequality* $\sum_{j=1}^{n} [\underline{a}_j, \overline{a}_j] x_j \geq [\underline{b}, \overline{b}]$*, the characteristic formula of the inequality is represented by* $\sum_{j=1}^{n} a_j x_j \geq b$ *such that* $a_j \in [\underline{a}_j, \overline{a}_j], \quad b \in [\underline{b}, \overline{b}]$.

Definition 6. *For each constraint inequality* $\sum_{j=1}^{n} [\underline{a}_j, \overline{a}_j] x_j \geq [\underline{b}, \overline{b}]$*, if there exists one characteristic formula such that its set of solution is the same as* $\overline{D}$ *or* $\underline{D}$*, then this characteristic formula is called as maximum value range inequality or minimum value range inequality.*

3.2 Fuzzy Number Linear Programming problems

A Fuzzy number Linear Programming (FLP) problem could be defined as follows:

$$\begin{aligned} \min \quad & Z(x) = \sum_{j=1}^{n} \tilde{c}_j x_j, \\ s.t. \quad & \sum_{j=1}^{n} \tilde{a}_{ij} x_j \geq \tilde{b}_i, \quad i = 1, \ldots, m, \\ & x_j \geq 0, \quad j = 1, \ldots, n, \end{aligned} \tag{3}$$

where $\tilde{c}_j$, $\tilde{a}_{ij}$, $\tilde{b}_i \in F(\mathbb{R})$ for all i's and j's, and the inequality relations are the standard ordering relation on $F(\mathbb{R})$.
According to Remark 1, we can denote the α-cuts of objective coefficients, technological coefficients and right hand side values as follows:

$$\tilde{c}_{j_\alpha} = [\underline{c}_j, \overline{c}_j] = [\alpha c_j^m + (1-\alpha) c_j^l, \alpha c_j^m + (1-\alpha) c_j^u],$$

$$\tilde{a}_{ij_\alpha} = [\underline{a}_{ij}, \overline{a}_{ij}] = [\alpha a_{ij}^m + (1-\alpha) a_{ij}^l, \alpha a_{ij}^m + (1-\alpha) a_{ij}^u],$$

$$\tilde{b}_{i_\alpha} = [\underline{b}_i, \overline{b}_i] = [\alpha b_i^m + (1-\alpha) b_i^l, \alpha b_i^m + (1-\alpha) b_i^u],$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Now by substituting these values in Problem (3), it is converted to an interval programming which can be solved by some convenient methods. By introducing α-cuts of objective function and constraints, the following model is obtained:

$$\begin{aligned}
\min \quad & Z(x)=\sum_{j=1}^{n}(\alpha c_j^m+(1-\alpha)c_j^l)x_j+\sum_{j=1}^{n}p_j(1-\alpha)(c_j^u-c_j^l),\\
s.t. \quad & \sum_{j=1}^{n}(\alpha a_{ij}^m+(1-\alpha)a_{ij}^l)x_j+\sum_{j=1}^{n}p_{ij}(1-\alpha)(a_{ij}^u-a_{ij}^l)\\
& \qquad -\beta_i(1-\alpha)(b_i^u-b_i^l)\ge \alpha b_i^m+(1-\alpha)b_i^l,\quad i=1,\dots,m,\\
& 0\le p_j\le x_j,\quad j=1,\dots,n,\\
& 0\le p_{ij}\le x_j,\quad j=1,\dots,n,\quad i=1,\dots,m,\\
& x_j\ge 0,\quad j=1,\dots,n,\\
& 0\le \beta_i\le 1,\quad i=1,\dots,m,
\end{aligned}\tag{4}$$

in which for different levels of $\alpha\in[0,1]$, we can reach a series of objective optimal values $Z_*^{(\alpha)}$.

Theorem 1. *If* $0\le\alpha_1\le\alpha_2\le1$*, then we have*

$$[Z_1^{(\alpha_1)},Z_2^{(\alpha_1)}]\supseteq[Z_1^{(\alpha_2)},Z_2^{(\alpha_2)}].$$

Proof. In Problem (3), if we consider the α- cuts of all coefficients of the problem, then the problem is converted to an interval programming in which we suppose that

$$\begin{aligned}
\underline{c}_j^r&=\alpha_r c_j^m+(1-\alpha_r)c_j^l, & \overline{c}_j^r&=\alpha_r c_j^m+(1-\alpha_r)c_j^u, & r&=1,2\ ,\\
\underline{a}_{ij}^r&=\alpha_r a_{ij}^m+(1-\alpha_r)a_{ij}^l, & \overline{a}_{ij}^r&=\alpha_r a_{ij}^m+(1-\alpha_r)a_{ij}^u, & r&=1,2\ ,\\
\underline{b}_i^r&=\alpha_r b_i^m+(1-\alpha_r)b_i^l, & \overline{b}_i^r&=\alpha_r b_i^m+(1-\alpha_r)b_i^u, & r&=1,2\ .
\end{aligned}$$

Considering interval coefficients, Problem (3) could be reduced into the following two classical linear programmings by Tong Shaocheng model Shaocheng (1994).

$$\begin{array}{ll}
\min\ \ Z_1^{(\alpha_r)}(x)=\sum_{j=1}^n \underline{c}_j^r x_j, & \min\ \ Z_2^{(\alpha_r)}(x)=\sum_{j=1}^n \overline{c}_j^r x_j,\\
s.t. & s.t.\\
\sum_{j=1}^n \overline{a}_{ij}^r x_j\ge \underline{b}_i^r,\quad i=1,\dots,m, & \sum_{j=1}^n \underline{a}_{ij}^r x_j\ge \overline{b}_i^r,\quad i=1,\dots,m,\\
x_j\ge0,\quad j=1,\dots,n, & x_j\ge0,\quad j=1,\dots,n.\\
(I) & (II)
\end{array}$$

Let $[Z_1^{(\alpha_1)},Z_2^{(\alpha_1)}]$ and $[Z_1^{(\alpha_2)},Z_2^{(\alpha_2)}]$ be the optimal interval values obtained from the LP problems above. Accoarding to Lemma 1, applying two fuzzy cuts $\alpha_1\le\alpha_2$ for fuzzy coefficients of (3), the relations below will be deduced:

$$[\underline{b}_i^2,\overline{b}_i^2]\subseteq[\underline{b}_i^1,\overline{b}_i^1],\qquad [\underline{a}_{ij}^2,\overline{a}_{ij}^2]\subseteq[\underline{a}_{ij}^1,\overline{a}_{ij}^1],\qquad [\underline{c}_j^2,\overline{c}_j^2]\subseteq[\underline{c}_j^1,\overline{c}_j^1].$$

In order to prove that $[Z_1^{(\alpha_1)},Z_2^{(\alpha_1)}]\supseteq[Z_1^{(\alpha_2)},Z_2^{(\alpha_2)}]$, we just need to show that $Z_1^{(\alpha_1)}\le Z_1^{(\alpha_2)}$ and $Z_2^{(\alpha_2)}\le Z_2^{(\alpha_1)}$.

First, we establish that the feasible region of Model (I) for the case $r=1$, is bigger than the case $r=2$. Suppose x° is a feasible solution to Model (I) for $r=2$. Then, we have

$$\begin{aligned}
&\overline{a}_{ij}^1\ge\underline{a}_{ij}^2\ ,\quad x_j^\circ\ge0\ ,\quad \forall i,j\Rightarrow \overline{a}_{ij}^1x_j^\circ\ge\underline{a}_{ij}^2x_j^\circ,\quad\forall i,j\ ,\\
&\qquad\qquad\qquad\qquad\qquad \Rightarrow\ \sum_{j=1}^n\overline{a}_{ij}^1x_j^\circ\ge\sum_{j=1}^n\underline{a}_{ij}^2x_j^\circ,\ \forall i.
\end{aligned}$$

From constraints of Model (I), for $r = 2$, we have $\sum_{j=1}^{n} \overline{a}_{ij}^{2} x_j^{\circ} \geq \underline{b}_i^2$, and on the other hand, $\underline{b}_i^2 \geq \underline{b}_i^1$, thus

$$\sum_{j=1}^{n} \overline{a}_{ij}^{1} x_j^{\circ} \geq \sum_{j=1}^{n} \underline{a}_{ij}^{2} x_j^{\circ} \geqslant \underline{b}_i^2 \geqslant \underline{b}_i^1.$$

Now, let x^* be the optimal solution of model (I) for $r = 2$. From the properties of α- cuts, we know $\underline{c}_j^2 \geq \underline{c}_j^1$. So, we have

$$\underline{c}_j^2 \geq \underline{c}_j^1 \ , \quad x_j^* \geq 0 \ , \quad \forall j \Rightarrow \quad \underline{c}_j^2 x_j^* \geq \underline{c}_j^1 x_j^*, \quad \forall j \ ,$$
$$\Rightarrow \sum_{j=1}^{n} \underline{c}_j^2 x_j^* \geq \sum_{j=1}^{n} \underline{c}_j^1 x_j^*, \Rightarrow \quad Z_1^{(\alpha_2)} \geq Z_1^{*(\alpha_1)} = \textstyle\sum_{j=1}^{n} \underline{c}_j^1 x_j^*.$$

As x^* is a feasible solution of model (I) for $r = 1$, thus $Z_1^{(\alpha_1)} \leq Z_1^{*(\alpha_1)}$. So, we have $Z_1^{(\alpha_2)} \geq Z_1^{(\alpha_1)}$. Similarly, it could be stablished that $Z_2^{(\alpha_1)} \geq Z_2^{(\alpha_2)}$.
We can generally say that, if $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 1$, then

$$[Z_1^{(\alpha_1)}, Z_2^{(\alpha_1)}] \supseteq [Z_1^{(\alpha_2)}, Z_2^{(\alpha_2)}] \supseteq \cdots \supseteq [Z_1^{(\alpha_n)}, Z_2^{(\alpha_n)}].$$

□

Corollary 1. *It is easy to establish that if we take different levels, say,* $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq 1$, *we can get a series of dynamic optimal intervals:*

$$[Z_1^{(\alpha_1)}, Z_2^{(\alpha_1)}] \supseteq [Z_1^{(\alpha_2)}, Z_2^{(\alpha_2)}] \supseteq \cdots \supseteq [Z_1^{(\alpha_n)}, Z_2^{(\alpha_n)}],$$

which give much information for decision maker. Besides, in the course of finding the best solution, considering $Z_1^{(\alpha)}$ *as the optimal solution and the assumptions above, we have a series of objective function optimal values as follows:*

$$Z_1^{(\alpha_1)} \leq Z_1^{(\alpha_2)} \leq \cdots \leq Z_1^{(\alpha_n)}$$

4 An application of improved approach in fractional programming

In this section, after giving the necessary preliminaries and a summary of the Charnes and Cooper Charnes & Cooper (1962) approach, we are going to apply our proposed improved approach to the linear fractional programming with interval numbers and propose an approach based on our improved approach previously mentioned. Besides, we will extend the approach to the linear fractional programming with fuzzy numbers. In the end of this section, with the help of Chakraborty and Gupta's approach Chakraborty & Gupta (2002) and by applying our improved approach, discussed in this study, we propose an algorithm to solve a multi-objective linear fractional programming problem with fuzzy parameters.

4.1 Interval Linear Fractional and Fuzzy Linear Fractional Programming problems

We define the general form of a Linear Fractional Programming (LFP) problem as follows:

$$\begin{aligned} \max \qquad & Z(x) = \frac{\sum_{j=1}^{n} c_j x_j + p}{\sum_{j=1}^{n} d_j x_j + q} = \frac{N(x)}{D(x)}, \\ s.t. \qquad & x \in S, \end{aligned} \tag{5}$$

where $S = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c, d \in \mathbb{R}^n$ and $p, q \in \mathbb{R}$. Problem (5) is said to be a standard concave-convex programming problem, when $N(x)$ is concave on S with $N(\eta) \geq 0$, for some $\eta \in S$ and $D(x)$ is convex and positive on S.
To evade the cases that for some values of x, $D(x)$ may be equal to zero, we should have either $\{x \geq 0, Ax \leq b \Rightarrow D(x) > 0\}$ or $\{x \geq 0, Ax \leq b \Rightarrow D(x) < 0\}$. For the sake of simplicity we suppose that LFP problem (5) satisfies the following condition:

$$x \geq 0, Ax \leq b \Rightarrow D(x) > 0. \tag{6}$$

Definition 7. *Craven (1988). Consider the two mathematical programming problems below:*

$$\begin{array}{llll} \max & F(x) & \max & G(x) \\ s.t. & x \in S \quad (i), & s.t. & x \in Q. \quad (ii) \end{array}$$

Two problems (i) and (ii) are equivalent if and only if there is a one-to-one mapping f of the feasible set of (i) onto the feasible set of (ii) such that $F(x) = G(f(x))$ for all $x \in S$.

Theorem 2. *Charnes & Cooper (1962). Suppose that no point $(y, 0)$ with $y \geq 0$ is feasible for the following linear programming problem:*

$$\begin{array}{ll} \max & c^T y + pt, \\ s.t. & d^T y + qt = 1, \\ & Ay - bt \leq 0, \\ & t \geq 0,\ y \geq 0,\ y \in \mathbb{R}^n,\ t \in \mathbb{R}. \end{array} \tag{7}$$

Assuming condition (6), then the LFP problem (5) is equivalent to the LP problem (7).

Now consider the two modified problems:

$$\begin{array}{ll} \max & tN(y/t), \\ s.t. & A(y/t) - b \leq 0, \\ & tD(y/t) = 1, \\ & t > 0,\ y \geq 0, \end{array} \tag{8}$$

and

$$\begin{array}{ll} \max & tN(y/t), \\ s.t. & A(y/t) - b \leq 0, \\ & tD(y/t) \leq 1, \\ & t > 0,\ y \geq 0, \end{array} \tag{9}$$

where (8) is reached from (5) by the substitutions $t = 1/D(x)$, $y = tx$ and (9) is deduced from the replacement of the equality constraint $tD(y/t) = 1$ by an inequality constraint $tD(y/t) \leq 1$ in Model (8).

Theorem 3. *Schaible (1976). For some $\eta \in S$, $N(\eta) \geq 0$, if (5) reaches a (global) maxmimum at $x = x^*$, then (9) reaches a (global) maximum at a point $(t, y) = (t^*, y^*)$, where $y^*/t^* = x^*$ and the objective functions are equal at these points.*

Theorem 4. *Schaible (1976). Suppose that (5) is a standard concave-covex programming problem that reaches a (global) maximum at a point x^*. Then the related modified problem (9) reaches the same maximum value at a point (t^*, y^*), where $y^*/t^* = x^*$. In addition, (9) has a concave objective function and a convex feasible set.*

For the problem:

$$\begin{aligned} \max \quad & Z(x) = \frac{N(x)}{D(x)}, \\ s.t. \quad & x \in S, \end{aligned} \tag{10}$$

where $N(x)$ is concave and negative for each $x \in S$, and $D(x)$ is concave and positive on S, we have the equivalences below:

$$\max_{x \in S} \frac{N(x)}{D(x)} \Leftrightarrow \min_{x \in S} \frac{-N(x)}{D(x)} \Leftrightarrow \max_{x \in S} \frac{D(x)}{-N(x)} \ ,$$

where $-N(x)$ is convex and positive. So, problem (10) is converted into a standard concave-convex programming problem. Therefore, by using Theorem 2, problem (10) can be transformed into the following LP problem:

$$\begin{aligned} \max \quad & tD(y/t), \\ s.t. \quad & A(y/t) - b \leq 0, \\ & -tN(y/t) \leq 1, \\ & t > 0, \ y \geq 0. \end{aligned} \tag{11}$$

4.1.1 The proposed approach for Interval Linear Fractional Programming problems

We define a linear fractional programming problem with interval numbers as follows:

$$\begin{aligned} \max \quad & Z(x) = \frac{\sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j] x_j + [\underline{p}, \overline{p}]}{\sum_{j=1}^{n} [\underline{d}_j, \overline{d}_j] x_j + [\underline{q}, \overline{q}]} \ , \\ s.t. \quad & \sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}] x_j \leq [\underline{b}_i, \overline{b}_i], \quad i = 1, \ldots, m, \\ & x_j \geq 0, \quad j = 1, \ldots, n. \end{aligned} \tag{12}$$

Accoarding to Charnes and Cooper Charnes & Cooper (1962) approach and by Theorem 2, we can consider the equivalent problem below:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j] y_j + [\underline{p}, \overline{p}] t, \\ s.t. \quad & \sum_{j=1}^{n} [\underline{d}_j, \overline{d}_j] y_j + [\underline{q}, \overline{q}] t = 1, \\ & \sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}] y_j - [\underline{b}_i, \overline{b}_i] t \leq 0, \quad i = 1, \ldots, m, \\ & t > 0, \ y \geq 0, \ y \in \mathbb{R}^n, \ t \in \mathbb{R}, \end{aligned} \tag{13}$$

where $y = tx$.
Now we have an ILP problem. By applying the given approach same in (4), we have the following

linear programming problem (4):

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} \underline{c}_j y_j + \sum_{j=1}^{n} p_j(\overline{c}_j - \underline{c}_j) + \underline{p}t + \lambda(\overline{p} - \underline{p}), \\
s.t. \quad & \sum_{j=1}^{n} \underline{d}_j y_j + \sum_{j=1}^{n} D_j(\overline{d}_j - \underline{d}_j) + \underline{q}t + \delta(\overline{q} - \underline{q}) = 1, \\
& \sum_{j=1}^{n} \underline{a}_{ij} y_j + \sum_{j=1}^{n} p_{ij}(\overline{a}_{ij} - \underline{a}_{ij}) - \underline{b}_i t - \beta_i(\overline{b}_i - \underline{b}_i) \leq 0, i = 1, \dots, m, \\
& 0 \leq p_j, D_j \leq y_j, \quad j = 1, \dots, n, \\
& 0 \leq p_{ij} \leq y_j, \quad j = 1, \dots, n, \quad i = 1, \dots, m, \\
& 0 \leq \beta_i, \delta, \lambda \leq t, \quad i = 1, \dots, m, \\
& t > 0, \ y \geq 0, \ y \in \mathbb{R}^n, \ t \in \mathbb{R}.
\end{aligned} \tag{14}
$$

4.1.2 The proposed approach for Fuzzy Linear Fractional Programming problems

Consider the following general form of a linear fractional programming problem with fuzzy numbers:

$$
\begin{aligned}
\max \quad & \tilde{Z}(x) = \frac{\sum_{j=1}^{n} \tilde{c}_j x_j + \tilde{p}}{\sum_{j=1}^{n} \tilde{d}_j x_j + \tilde{q}}, \\
s.t. \quad & \sum_{j=1}^{n} \tilde{a_{ij}} x_j \leq \tilde{b_i}, \quad i = 1, \dots, m, \\
& x_j \geq 0, \quad j = 1, \dots, n.
\end{aligned} \tag{15}
$$

As we have already seen, accoarding to Charnes and Cooper approach and by Theorem 2, we can consider the equivalent problem below:

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} \tilde{c}_j y_j + \tilde{p}t, \\
s.t. \quad & \sum_{j=1}^{n} \tilde{d}_j y_j + \tilde{q}t = 1, \\
& \sum_{j=1}^{n} \tilde{a_{ij}} y_j - \tilde{b_i} t \leq 0, \quad i = 1, \dots, m, \\
& t > 0, \ y \geq 0, \ y \in \mathbb{R}^n, \ t \in \mathbb{R}.
\end{aligned} \tag{16}
$$

Introducing α-cuts of objective function and constraints and following model (4), these can lead to the following model:

$$\begin{aligned}
\max \quad & \sum_{j=1}^{n}(\alpha c_j^m+(1-\alpha)c_j^l)y_j+\sum_{j=1}^{n}p_j(1-\alpha)(c_j^u-c_j^l)+(\alpha p^m+(1-\alpha)p^l)t+\lambda(1-\alpha)(p^u-p^l),\\
s.t. \quad & \sum_{j=1}^{n}(\alpha d_j^m+(1-\alpha)d_j^l)y_j+\sum_{j=1}^{n}D_j(1-\alpha)(d_j^u-d_j^l)+(\alpha q^m+(1-\alpha)q^l)t+\delta(1-\alpha)(q^u-q^l)=1,\\
& \sum_{j=1}^{n}(\alpha a_{ij}^m+(1-\alpha)a_{ij}^l)y_j+\sum_{j=1}^{n}p_{ij}(1-\alpha)(a_{ij}^u-a_{ij}^l)-(\alpha b_i^m+(1-\alpha)b_i^l)t-\beta_i(1-\alpha)(b_i^u-b_i^l)\geq 0,\\
& \qquad\qquad i=1,\ldots,m,\\
& 0\leq p_j, D_j\leq y_j, \quad j=1,\ldots,n,\\
& 0\leq p_{ij}\leq y_j, \quad j=1,\ldots,n, \quad i=1,\ldots,m,\\
& 0\leq \beta_i,\delta,\lambda\leq t, \quad i=1,\ldots,m,\\
& t>0,\ y\geq 0,\ y\in\mathbb{R}^n,\ t\in\mathbb{R}, \qquad (17)
\end{aligned}$$

in which for different levels of $\alpha \in [0,1]$, we can obtaine a series of objective optimal values.

Now we deal with a simple economic application of the considered interval linear fractional programming problem model in a real life situation and use our proposed approach to solve it.

4.2 The proposed algorithm for solving a Fuzzy Multi-Objective Linear Fractional Programming problem

Consider the general form of a Multi-Objective Linear Fractional Programming problem with fuzzy parameters (abbreviated as FMOLFP problem) as follows Chakraborty & Gupta (2002):

$$\begin{aligned}
\max \quad & \tilde{Z}(x)=(\tilde{Z}_1(x),\tilde{Z}_2(x),\ldots,\tilde{Z}_k(x)),\\
s.t. \quad & x\in\tilde{S}, \qquad (18)
\end{aligned}$$

where $\tilde{Z}_l(x)=\frac{\sum_{j=1}^{n}\tilde{c}_{lj}x_j+\tilde{p}_l}{\sum_{j=1}^{n}\tilde{d}_{lj}x_j+\tilde{q}_l}=\frac{\tilde{N}_l(x)}{\tilde{D}_l(x)}$, $\tilde{c}_l,\tilde{d}_l\in F^n(\mathbb{R})$, $\tilde{p}_l,\tilde{q}_l\in F(\mathbb{R})$ for all $l=1,\ldots,k$ and $\tilde{S}=\{x\in\mathbb{R}^n:\tilde{A}x\leq\tilde{b},x\geq 0\}$, $\tilde{A}\in F^{m\times n}(\mathbb{R})$ and $\tilde{b}\in F^m(\mathbb{R})$.

Consider the two index sets below

$$L=\{l:\tilde{N}_{l_\alpha}(x)\geq_\alpha 0 \ \textit{for some}\ x\in\tilde{S}\}, \qquad L^c=\{l:\tilde{N}_{l_\alpha}(x)<_\alpha 0\ \textit{for each}\ x\in\tilde{S}\},$$

where $L\cup L^c=\{1,2,\ldots,k\}$. Let $\tilde{D}_\alpha(.)$ be positive on $\tilde{S}$, where $\tilde{S}$ is non-empty and bounded. Let t stands for the least value of $\frac{1}{\tilde{d}_l x+\tilde{q}_l}$ and $\frac{-1}{\tilde{c}_l x+\tilde{p}_l}$ for $l\in L$ and $l\in L^c$, respectively. Then $t\leq\frac{1}{\tilde{d}_l x+\tilde{q}_l}$ for $l\in L$ and $t\leq\frac{-1}{\tilde{c}_l x+\tilde{p}_l}$ for $l\in L^c$. By the transformation $y=tx\,(t>0)$, Theorem 2, Theorem 3, and also using the mentioned inequalities and considering an specific α (to linearize, follow model (17)), an equivalent Multi-Objective Linear Programming problem for FMOLFP problem can be written as follows:

$$\begin{aligned}
\max \quad & \{t\tilde{N}_{l_\alpha}(\frac{y}{t})\ \textit{for}\ l\in L\ ;\ t\tilde{D}_{l_\alpha}(\frac{y}{t})\ \textit{for}\ l\in L^c\},\\
s.t. \quad & t\tilde{D}_{l_\alpha}(\frac{y}{t})\leq 1, \quad \textit{for}\ l\in L,\\
& -t\tilde{N}_{l_\alpha}(\frac{y}{t})\leq 1, \quad \textit{for}\ l\in L^c,\\
& \tilde{A}_\alpha(\frac{y}{t})-\tilde{b}_\alpha\leq 0,\\
& y\geq 0, \quad t>0. \qquad (19)
\end{aligned}$$

We note that the feasible region of problem (19) is always non-empty convex set (for proving, we refer to Chakraborty & Gupta (2002)).

Considering the α-cut of the coefficients, we define the membership function of each objective function as follows:

$$\mu_l^{(\alpha)}(t\tilde{N}_{l_\alpha}(\frac{y}{t})) = \begin{cases} 0 & , \ t\tilde{N}_{l_\alpha}(\frac{y}{t}) \leq 0 \\ \frac{t\tilde{N}_{l_\alpha}(\frac{y}{t})-0}{\bar{Z}_l-0} & , \ 0 < t\tilde{N}_{l_\alpha}(\frac{y}{t}) < \bar{Z}_l \\ 1 & , \ t\tilde{N}_{l_\alpha}(\frac{y}{t}) \geq \bar{Z}_l, \end{cases} \tag{20}$$

and

$$\mu_l^{(\alpha)}(t\tilde{D}_{l_\alpha}(\frac{y}{t})) = \begin{cases} 0 & , \ t\tilde{D}_{l_\alpha}(\frac{y}{t}) \leq 0 \\ \frac{t\tilde{D}_{l_\alpha}(\frac{y}{t})-0}{\bar{Z}_l-0} & , \ 0 < t\tilde{D}_{l_\alpha}(\frac{y}{t}) < \bar{Z}_l \\ 1 & , \ t\tilde{D}_{l_\alpha}(\frac{y}{t}) \geq \bar{Z}_l, \end{cases} \tag{21}$$

for $l \in L$ and $l \in L^c$, respectively.
Now, via the membership functions (20) and (21), and using Zimmermann's min operator Zimmermann (1975), the MOLP problem (19) can be converted to the crisp model below

$$\begin{aligned} \max \quad & \nu \\ s.t. \quad & \mu_l^{(\alpha)}(t\tilde{N}_{l_\alpha}(\frac{y}{t})) \geq \nu, \quad l \in L \\ & \mu_l^{(\alpha)}(t\tilde{D}_{l_\alpha}(\frac{y}{t})) \geq \nu, \quad l \in L^c \\ & t\tilde{D}_{l_\alpha}(\frac{y}{t}) \leq 1, \quad l \in L \\ & -t\tilde{N}_{l_\alpha}(\frac{y}{t}) \leq 1, \quad l \in L^c \\ & \tilde{A}_\alpha(y) - \tilde{b}_\alpha t \leq 0, \\ & y \geq 0, \quad t > 0. \end{aligned} \tag{22}$$

Remark 2. *In the proposed method, it is supposed that the nature of $\tilde{N}_{l_\alpha}(x)$ for all $l = 1, 2, \ldots, k$ are determinate, this means that the index set L and L^c are known. If they are not determinate clearly but it is known that the denominators are positive in the feasible region, then to discover the index sets L, L^c and the maximum aspiration levels $\bar{Z}_l$, we proceed as follows:*

1. *For a given α, maximize each objective function $\tilde{Z}_l(x)$ subject to the given set of constraints by the use of model (17). Suppose that $Z^*_{l_\alpha}$ is the maximum value of $\tilde{Z}_l(x)$ for $l = 1, 2, \ldots, k$.*
2. *If $Z^*_{l_\alpha}$ is nonnegative, then $l \in L$, and if $Z^*_{l_\alpha}$ is negative, then $l \in L^c$.*
3. *If $l \in L$, then $\bar{Z}_l = Z^*_{l_\alpha}$ and if $l \in L^c$, then $\bar{Z}_l = \frac{-1}{Z^*_{l_\alpha}}$.*

Here, we give a general algorithm to obtain a solution to the FMOLFP problem (18). The steps of the proposed algorithm can be synthesized as follows:

Algorithm 4.1 (*main steps of the proposed algorithm*)

Step 1: Specify the α-cuts for the coefficients of each objective function and constraints of the problem (18). For the specified α, go to step 2.

Step 2: Using model (17), and obtain the best solution to each objective function. Let Z^*_l be the obtained optimal value in this step for each l.

Step 3: Define the membership function for each objective function helping (20) and (21).

Step 4: Solve the model (22).

Step 5: Find the optimal solution x^* using the values y and t in step 4, and find the optimal values $Z^*_{l_\alpha}$ for the original problem (18).

Remark 3. *It is worth mentioning that for different levels of* $\alpha \in [0,1]$*, one can reach a series of objective optimal values* $Z^*_{l_\alpha}$ *and also if both the best solution and the worth solution for each objective function are found, one can reach the interval optimal solutions which provide much information for him/her.*

The following numerical example illustrates our approach.

Example 1. *Consider the following FMOLFP problem*

$$\begin{aligned} \max \quad & Z(x) = \left(\frac{\tilde{1}x_1 + \tilde{1}x_2}{\tilde{2}x_1 + \tilde{1}x_2 + \tilde{1}}, \frac{\tilde{4}x_1 + \tilde{3}x_2}{\tilde{6}x_1 + \tilde{2}x_2 + \tilde{1}}, \frac{\tilde{2}x_1 + \tilde{4}x_2 + \tilde{1}}{\tilde{1}x_1 + \tilde{2}x_2 + \tilde{3}}\right), \\ s.t. \quad & \tilde{2}x_1 - \tilde{1}x_2 \geq \tilde{1}, \\ & \tilde{1}x_1 + \tilde{4}x_2 \leq \tilde{18}, \\ & \tilde{2}x_1 + \tilde{4}x_2 \geq \tilde{10}, \\ & \tilde{1}x_1 \geq \tilde{4}, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{23}$$

The memebership function of each objective function for the above model can be written as follows:

$$\mu_l^{(\alpha)}(f_l(y,t)) = \begin{cases} 0 & , \ f_l(y,t) \leq 0 \\ \frac{ay_1+by_2+ct}{Z^*_{l_\alpha}} & , \ 0 < f_l(y,t) < Z^*_{l_\alpha} \\ 1 & , \ f_l(y,t) \geq Z^*_{l_\alpha}, \end{cases} \tag{24}$$

where the value of parameters have been collected in Table 1.
The results of applying Model (22) and for the original Problem (23) are summarized in Table 2.

Table 1: The value of parameters in (24).

	l=1				l=2				l=3			
α	a	b	c	$Z^*_{l_\alpha}$	a	b	c	$Z^*_{l_\alpha}$	a	b	c	$Z^*_{l_\alpha}$
0	$\frac{3}{2}$	$\frac{3}{2}$	0	1.53	5	$\frac{7}{2}$	0	1.362	$\frac{5}{2}$	5	$\frac{3}{2}$	4.405
0.25	$\frac{11}{8}$	$\frac{11}{8}$	0	1.2	$\frac{19}{4}$	$\frac{27}{8}$	0	1.207	$\frac{19}{8}$	$\frac{19}{4}$	$\frac{11}{8}$	3.35
0.5	$\frac{5}{4}$	$\frac{5}{4}$	0	0.95	$\frac{9}{2}$	$\frac{13}{4}$	0	1.064	$\frac{9}{4}$	$\frac{9}{2}$	$\frac{5}{4}$	2.645
0.75	$\frac{9}{8}$	$\frac{9}{8}$	0	0.755	$\frac{17}{4}$	$\frac{25}{8}$	0	0.939	$\frac{17}{8}$	$\frac{17}{4}$	$\frac{9}{8}$	2.14

Table 2: The results of problem parameters (22) and (23).

α	ν^*	y_1	y_2	t	x_1^*	x_2^*	$Z^*_{1_\alpha}$	$Z^*_{2_\alpha}$	$Z^*_{3_\alpha}$
0	0.309	0.139	0.191	0.0401	3.466	4.763	0.648	0.899	1.687
0.25	0.348	0.133	0.17	0.0385	3.454	4.415	0.639	0.886	1.673
0.50	0.365	0.13	0.147	0.0361	3.601	4.072	0.625	0.866	1.661
0.75	0.379	0.1272	0.1275	0.0337	3.774	3.783	0.613	0.847	1.651
1	0.391	0.125	0.109	0.031	4.03	3.52	0.6	0.828	1.64

5 Conclusion

Uncertainty and inaccuracy of data are inseparable from the many real-world phenomena. Therefore, interval and fuzzy mathematical programming are powerful alternative tackles to formulate these problems. There are some methods to solve them in the literature. In the present research, we have adopted the Tong Shaocheng method, the Saati et al approach, and the proposed method, which is based on the Tong Shaocheng method, and deal with their efficiency. After a comprehensive study of the methods, we have given some advantages, disadvantages, and results of the considered methods, and so an improved approach has been obtained. Moreover, we have applied our improved approach to fractional programming, especially to fuzzy multi-objective linear fractional programming. One of the advantages of our research is that the necessary proposed theorems, lemmas, propositions, and claims have been proved.

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